ON A PROBLEM OF EQUILIBRIUM OF ELASTIC HALF-SPACE

(OB ODNOI ZADACHE RAVNOVESIIA UPRUGOGO POLUPROSTRANSTVA)

PMM Vol.23, No.6, 1959, pp. 1129

I.Sh. RABINOVICH
(Leningrad)

(Received 13 July 1959)

The problem regarding the action, on the boundary of a half-space along an infinite straight line, of normal loading varying in accordance with a linear law, is considered.

Let the boundary of elastic half-space be the plane $x = 0$, the x-axis being directed along the internal normal. The loading is distributed along the line $y = 0$ in accordance with the law

$$p_x = p_0 x \tag{1}$$

At the point ($x = \xi$, $y = z = 0$) we isolate from the distributed loading the elementary force $dP_\xi = p_\xi d\xi = p_0 \xi \, d\xi$. The components of elementary displacement produced by this force, on the basis of Boussinesq's solution ([1], p. 157) are

$$du_\xi = \frac{p_0}{4\pi\mu}\frac{(x-\xi)\,z\xi d\xi}{R_1{}^3} - \frac{p_0}{4\pi(\lambda-\mu)}\frac{(x-\xi)\,\xi d\xi}{R_1(R_1+z)} \qquad (R_1{}^2 = (x-\xi)^2 + y^2 + z^2) \tag{2}$$

$$dv_\xi = \frac{p_0}{4\pi\mu}\frac{yz\xi d\xi}{R_1{}^3} - \frac{p_0}{4\pi(\lambda+\mu)}\frac{y\xi d\xi}{R_1(R_1+z)}, \qquad dw_\xi = \frac{p_0}{4\pi\mu}\frac{z^2\xi d\xi}{R_1{}^3} + \frac{p_0(\lambda+2\mu)}{4\pi\mu(\lambda+\mu)}\frac{\xi d\xi}{R_1}$$

Integrating expressions (2) in the range of variation of ξ form $-\infty$ to $+\infty$, we obtain the solution of the formulated problem in terms of components of displacements:

$$u = \frac{p_0(\lambda+2\mu)}{2\pi\mu(\lambda+\mu)}\, z \ln r + \frac{p_0}{2\pi(\lambda+\mu)}\, y \operatorname{arc\,tg}\frac{z}{y} + \frac{p_0 z}{2\pi\mu} \qquad (r^2 = y^2 + z^2)$$

$$v = \frac{p_0}{2\pi\mu}\frac{xyz}{r^2} + \frac{p_0}{2\pi(\lambda+\mu)}\, x \operatorname{arc\,tg}\frac{z}{y}, \qquad w = \frac{p_0}{2\pi\mu}\frac{xz^2}{r^2} - \frac{p_0(\lambda+2\mu)}{2\pi\mu(\lambda+\mu)}\, x \ln r \tag{3}$$

From here it is easy to obtain the relationship for the components of stress

$$X_x = \frac{2p_0\sigma}{\pi}\frac{xz}{r^2} \qquad X_y = \frac{p_0}{\pi}\frac{yz}{r^2} + \frac{p_0(1-2\sigma)}{\pi}\operatorname{arc\,tg}\frac{z}{y}, \qquad Y_y = -\frac{2p_0}{\pi}\frac{xy^2z}{r^4}$$

$$Y_z = -\frac{2p_0}{\pi}\frac{xyz^2}{r^4}, \qquad Z_z = -\frac{2p_0}{\pi}\frac{xz^3}{r^4} \qquad Z_x = \frac{p_0}{\pi}\frac{z^2}{r^2} + \frac{p_0(1-\sigma)}{\pi} \tag{4}$$

Here σ is Poisson's ratio.

The solution of Flamant for the case when the intensity of the distributed loading is equal to p_0, has the following form in our notation in terms of the components of displacement [1]

$$u^\circ = 0, \quad v^\circ = \frac{p_0}{2\pi\mu}\frac{yz}{r^2} - \frac{p_0}{2\pi(\lambda+\mu)} \operatorname{arc\,tg} \frac{y}{z}, \quad w^\circ = -\frac{p_0}{2\pi\mu}\frac{y^2}{r^2} - \frac{p_0(\lambda+2\mu)}{2\pi\mu(\lambda+\mu)} \ln r \quad (5)$$

By means of elementary transformations these expressions are reduced to the form

$$u^\circ = 0, \quad v^\circ = \frac{p_0}{2\pi\mu}\frac{yz}{r^2} + \frac{p_0}{2\pi(\lambda+\mu)} \operatorname{arc\,tg} \frac{z}{y}, \quad w^\circ = \frac{p_0}{2\pi\mu}\frac{z^2}{r^2} - \frac{p_0(\lambda+2\mu)}{2\pi\mu(\lambda+\mu)} \ln r \quad (6)$$

The following components of stress ([1], p. 207) correspond to these displacements

$$X_x{}^\circ = \frac{2p_0\sigma}{\pi}\frac{z}{r^2}, \quad Y_y{}^\circ = -\frac{2p_0}{\pi}\frac{zy^2}{r^4}, \quad Y_z{}^\circ = -\frac{2p_0}{\pi}\frac{yz^2}{r^4}, \quad Z_z{}^\circ = -\frac{2p_0}{\pi}\frac{z^3}{r^4} \quad (7)$$

Comparing the obtained solution (3) and (4) with the Flamant solution (6) and (7) we note the following:

$$v = v^\circ x, \quad w = w^\circ x, \quad X_x = X_x{}^\circ x, \quad Y_y = Y_y{}^\circ x, \quad Z_z = Z_z{}^\circ x \quad Y_z = Y_z{}^\circ x$$

The analogy between the two problems is not limited by this. In Flamant's problem $u^0 = 0$ indicates the presence of a plane state of strain. In our problem $u \neq 0$, but $du/dx = 0$, the linear deformation is absent along the x-axis, all planes $x = \text{const}$ are equally displaced.

It should be noted that the problem considered may be also be solved by the method of Gutman, suggested especially for problems with a linearly varying loading [2].

BIBLIOGRAPHY

1. Leibenzon, L.S., *Kurs teorii uprugosti (A Course on the Theory of Elasticity)*. 1947.

2. Gutman, S.G., Some problems of the statics of large dams in the light of laboratorial model research. *Extrait du Sixième Congrès de Grands Barrages*, New York, 1958; Paris, Imprimerie Gauthier Villars.

Translated by G.H.